2.5.2 Derivation of the Equation of Radiative Transport of Energy

This equation is written as a condition for the temperature gradient necessary for the required energy flow outwards in a stellar interior.

For radiative transport in stars, the mean free path $\ell_{\rm ph}$ of the "transporting particles" (= photons) is very small compared to the characteristic length R (= stellar radius) over which the transport extends: $\ell_{\rm ph}/R_\odot \approx 3 \times 10^{-11}$. We can therefore treat the radiative transport as a diffusion process and this produces an enormous simplification in deriving the equation of radiative transport.

From diffusion theory, the diffusive flux $\vec{\jmath}$ of particles (per unit area and time) between places with differing particle density n is given by

$$\vec{\jmath} = -D\nabla n \tag{2.6}$$

where D is the coefficient of diffusion which is given by:

$$D = \frac{1}{3} v \ell_{\rm p} \tag{2.7}$$

and is determined by the average values of mean velocity v and mean free path $\ell_{\rm p}$ of the particles.

For a stellar interior, to obtain the corresponding diffusive flux of radiative energy $\vec{F}$, we replace n by the energy density of radiation U where

$$U = aT^4, \tag{2.8}$$

v by the velocity of light c and $\ell_{\rm p}$ by $\ell_{\rm ph}$ where $\ell_{\rm ph} = 1/\kappa\rho$ i.e. the transporting particles are photons. Here, a is the radiation density constant.

Assuming spherical symmetry, $\vec{F}$ has only a radial component

$$F_r = |\vec{F}| = F(r) \tag{2.9}$$

and ∇U reduces to the derivative in the radial direction:

$$\frac{\partial U}{\partial r} = 4aT^3 \frac{\partial T}{\partial r}. \tag{2.10}$$

Then equations (2.6) and (2.7) become

$$F(r) = -\frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{\partial T}{\partial r} \approx -\frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{dT}{dr}. \tag{2.11}$$

Note that this equation can be interpreted formally as an equation for heat conduction by writing

$$\vec{F} = -k_{\rm rad} \nabla T \tag{2.12}$$

where

$$k_{\rm rad} = \frac{4ac}{3}\frac{T^3}{\kappa\rho} \tag{2.13}$$

which represents the coefficient of conduction for radiative transport.

We solve (2.11) for the temperature gradient and replace $F(r)$ by the luminosity

$$L(r) = 4\pi r^2 F(r) \tag{2.14}$$

then

$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho}{r^2T^3}\,L(r) \tag{2.15}$$

As the surface of the star is approached, this simple equation becomes invalid because the density decreases, and thus the mfp increases until it exceeds the remaining distance to the stellar surface. The diffuse approximation breaks down and we have to solve the equation of radiative transfer in the stellar atmosphere (Section 3).

2.5.3 The Rosseland Mean Opacity κ_R

The above equations are independent of frequency ν; F and L are integrated over all ν so that the opacity κ must represent a "proper mean" over ν. We can now derive the average form for this opacity. Since κ depends on ν, we add a subscript ν to frequency-dependent quantities:

$$\vec{F}_\nu = -D_\nu \nabla U_\nu \tag{2.16}$$

$$D_\nu = \frac{c}{3\kappa_\nu\rho} \tag{2.17}$$

and

$$U_\nu = \frac{4\pi}{c}B(\nu,T) \tag{2.18}$$

thus

$$\nabla U_\nu = \frac{4\pi}{c}\frac{\partial B}{\partial T}\nabla T \tag{2.19}$$

Inserting into (2.16) and integrating over all ν to obtain the total flux $\vec{F}$,

$$\vec{F} = -\left[\frac{4\pi}{3\rho}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}\,\mathrm{d}\nu\right]\nabla T. \tag{2.20}$$

We have thus regained (2.12) but with

$$k_{\rm rad} = \frac{4\pi}{3\rho}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}\,\mathrm{d}\nu. \tag{2.21}$$

Equating this expression for $k_{\rm rad}$ to that in the averaged form of (2.13), we have the proper formula for averaging the absorption coefficient:

$$\frac{1}{\kappa_R} = \frac{\pi}{acT^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}\,\mathrm{d}\nu. \tag{2.22}$$

Since

$$\int_0^\infty \frac{\partial B}{\partial T}\,\mathrm{d}\nu = \frac{acT^3}{\pi}, \tag{2.23}$$

then the formal equation for the Rosseland mean opacity k_R is given by

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B}{\partial T}\,\mathrm{d}\nu}. \tag{2.24}$$

We can now write the equation of radiative transport in the standard form using k_R. From (2.20) and (2.14),

$$L(r) = -\frac{16\pi^2 r^2}{3\rho(r)}\left[\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}\,\mathrm{d}\nu\right]\frac{dT}{dr} \tag{2.25}$$

$$= -\frac{16\pi^2 r^2}{3\rho(r)}\frac{1}{\kappa_R}\int_0^\infty \frac{\partial B}{\partial T}\,\mathrm{d}\nu \tag{2.26}$$

and with equation (2.23),

$$L(r) = \frac{-16\pi ac}{3k_R\rho(r)}\, r^2\, T^3\frac{dT}{dr} \tag{2.27}$$

or

$$\frac{dT}{dr} = -\frac{3k_R\rho(r)}{16\pi acr^2T^3(r)}L(r) \tag{2.28}$$

i.e. L(r) is proportional to the temperature gradient with a coefficient that depends on the properties of the matter and the temperature. This is the equation of radiative transport.