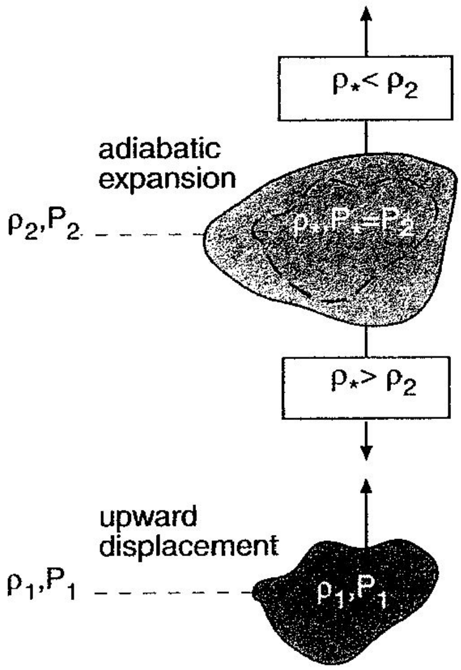
5. CONVECTION

Convection occurs in stars when the temperature gradient becomes very large. Cool stars, like the Sun, have outer convective zones whereas massive stars have a central convective zone because of the huge luminosities generated by their small cores.

5.1 Schwarzschild Criterion for Stability against Convection

How large does the temperature gradient have to be to cause convection and make the star unstable? Schwarzschild was the first to study this quantitatively (1906).



[From Prialnik (2000)]

To decide whether convection occurs in a star, we will consider a bubble or blob of material of volume V in pressure equilibrium with its surroundings. Initially both the bubble and its surroundings have temperature T_1, pressure P_1 and density ρ_1.

Suppose the temperature of the bubble is perturbed (e.g. through thermal motions) and it acquires a new temperature T_1'. If $T_1' > T_1$, and pressure balance is maintained (assuming time-scale for removing the pressure imbalance << time-scale to re-establish thermal equilibrium), then the equation of state ($P \propto \rho T$) demands $\rho_1' < \rho_1$.

Given a local gravity g, the bubble will feel a buoyancy force per unit volume:

$$f = -\Delta\rho\, g = \rho_1\, g - \rho_1'\, g$$

As a result, the bubble will rise a distance Δr to a location in the star where the surroundings have T_2, P_2 and ρ_2. As it rises, the bubble will expand adiabatically to a larger volume to maintain pressure equilibrium with its surroundings (since $P_2 < P_1$) and its new density will be ρ^*, which is not necessarily equal to its surroundings. The bubble's temperature may also not be equal to that of the surroundings.

The buoyancy force on the bubble will now be

$$\rho_2 \, g - \rho_* \, g$$

If $\rho_* < \rho_2$, the bubble will continue to rise and the star will be dynamically unstable to convection. If $\rho_* > \rho_2$ then the bubble will sink back to its original position.

Considering the change in the internal density of the bubble $\Delta\rho_i$ compared to the density gradient of the surroundings $\Delta\rho_s$ as it travels upwards a distance Δr, we can write that convection will not occur if:

$$\left|\frac{d\rho}{dr}\right|_i > \left|\frac{d\rho}{dr}\right|_s \tag{5.1}$$

where $i =$ indicates how the internal density of the bubble has changed with radius, and $s =$ indicates the density gradient in the surroundings.

Since we have assumed that the pressure of the bubble and its surroundings are the same, we can also write the stability criterion in terms of the temperature gradients ($\rho T =$const):

$$\left|\frac{dT}{dr}\right|_s < \left|\frac{dT}{dr}\right|_i$$

If we assume that the star is in radiative equilibrium, the temperature gradient of the surroundings will be the radiative temperature gradient $\left|\frac{dT}{dr}\right|_{\text{rad}}$. This must be less than the temperature gradient experienced by the bubble rising adiabatically i.e. the adiabatic temperature gradient $\left|\frac{dT}{dr}\right|_{\text{ad}}$. So

$$\left|\frac{dT}{dr}\right|_{\text{rad}} < \left|\frac{dT}{dr}\right|_{\text{ad}} \tag{5.2}$$

for stability against convection.

Thus if the radiative temperature gradient is less than than the adiabatic temperature gradient, convection will not occur and energy transport will be by radiation. If, however, the radiative temperature gradient becomes steeper, and is greater than or equal to the adiabatic temperature gradient, this region in a star will become convectively unstable and energy transport will be by convection. In other words, the star will adapt to the temperature gradient which is less steep, whether it is radiative or adiabatic.

Re-writing equation [5.2] in terms of dT/dP using

$$\frac{dT}{dr} = \frac{dT}{dP}\frac{dP}{dr}$$

we have

$$\left|\frac{dT}{dP}\right|_{\text{rad}} < \left|\frac{dT}{dP}\right|_{\text{ad}}$$

For an adiabatic gas where γ is the ratio of the specific heats,

$$P \propto \rho^\gamma \quad \text{and} \quad P \propto \rho T$$

or combining these two equations

$$T \propto \frac{P}{\rho} \propto \frac{\rho^\gamma}{\rho} \propto \rho^{\gamma-1} \propto \frac{P}{P^{1/\gamma}} \propto P^{\frac{\gamma-1}{\gamma}}$$

and then

$$\frac{dT}{dP}\frac{P}{T} = \frac{\gamma-1}{\gamma}.$$

Now

$$\frac{dT}{dP}\frac{P}{T} = \frac{d\ln T}{d\ln P}$$

where we have used the relation

$$\int \frac{dx}{x} = \ln x.$$

Thus

$$\left|\frac{d\ln T}{d\ln P}\right|_{\rm rad} < \left|\frac{d\ln T}{d\ln P}\right|_{\rm ad}$$

for stability.

Now

$$\left|\frac{d\ln T}{d\ln P}\right|_{\rm ad} = \frac{dT}{dP}\frac{P}{T} = \frac{\gamma-1}{\gamma}$$

and the stability criterion can be written

$$\left|\frac{d\ln T}{d\ln P}\right|_{\rm rad} < \frac{\gamma-1}{\gamma}.$$

The full Schwarzschild criterion for stability against convection is then given by

$$\left|\frac{d\ln T}{d\ln P}\right|_{\rm rad} < \left|\frac{d\ln T}{d\ln P}\right|_{\rm ad} < \frac{\gamma-1}{\gamma}. \tag{5.3}$$

Or in terms of the temperature gradient

$$\left|\frac{dT}{dr}\right|_{\rm ad} = \frac{dT}{dP}\frac{dP}{dr} = \frac{d\ln T}{d\ln P}\frac{T}{P}\frac{dP}{dr} = \frac{T}{P}\frac{dP}{dr}(\frac{\gamma-1}{\gamma})$$

Thus

$$\left|\frac{dT}{dr}\right|_{\rm rad} < \left|\frac{dT}{dr}\right|_{\rm ad} < \left|\left(\frac{\gamma-1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}\right| \tag{5.4}$$

for stability against convection.