Quantum Physics – Some part A type revision questions, mostly on the Schrödinger equation (with solutions)

A1. Let $\psi(x) = A(x-a)(x-b)$ the wavefunction of a particle which is confined to move freely in the one-dimensional interval $-2 \le x \le 2$ (in other words, the potential is $V = 0$ for $-2 \le x \le 2$ and $V = \infty$ for $x > 2$ and $x < -2$). Find a, b and A.

S1. The wavefunction must vanish at the boundary of the interval, i.e. we must impose $\psi(2) = \psi(-2) = 0$. Hence $a = 2$ and $b = -2$. Finally A is obtained from imposing $1 = \int_{-2}^{2} \psi(x)|^2$, i.e. we must require that

$$|A|^2 \int_{-2}^{2} (x-2)^2(x+2)^2 \;=\; 1 \; .$$

This gives (up to an irrelevant, overall phase) $A = (15/512)^{1/2}$.

A2. Consider a particle which can move in one dimension between $-\infty$ and $+\infty$. Explain why we need $\psi(x) \to 0$ as $x \to \pm\infty$.

S2. The wavefunction must be integrable i.e. we need to be able to impose that $\int_{-\infty}^{+\infty} |\psi(x)|^2 = 1$ which in particular implies that the integral must converge! A necessary condition for the integral to converge is precisely $\psi(x) \to 0$ as $x \to \pm\infty$.

A3. A single slit of length L is irradiated with light of wavelength λ. Explain under which conditions we observe diffraction.

S3. We need the wavelength to be comparable to the size of the slit, i.e. $\lambda \sim L$.

A4. Explain why the classical concept of "trajectory" of a particle does not make sense at the quantum level.

S4. Because of Heisenberg's uncertainty principle, we cannot measure to infinite accuracy the position and momentum of a particle at the same time. Hence we cannot define its trajectory! (knowing the trajectory of a particle precisely means knowing position and momentum of the particle at any time).

A5. Write down the Schrödinger equation for a single particle confined to move freely in the line interval $-L \le x \le L$ (in other words, the potential is $V = 0$ for $-L \le x \le L$ and $V = \infty$ for $x > L$ and $x < -L$).

S5.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \;=\; i\hbar\frac{\partial}{\partial t}\psi(x,t)\ ,$$

with the boundary condition $\psi(L,t) = \psi(-L,t) = 0$, and furthermore $\psi(x,t) = 0$ for $x < -L$ or $x > L$. Remember to write down the boundary conditions, which are part of the equation itself!

A6. Write down a one-dimensional plane wave as a function of p and E. What are the relations of p and E to ω and k?

S6. $\psi_{\text{free}}(x,t) = e^{\frac{i(px-Et)}{\hbar}}$. The relations are $p = \hbar k$ and $E = \hbar\omega$ are the two (de Broglie) relations. In terms of ω and k the plane wave has the form (familiar from optics) $e^{i(kx-\omega t)}$. Also remember that $T = 2\pi/\omega$ and $k = 2\pi/\lambda$.

A7. Check that the plane wave is a solution to the Schrödinger equation for a free non-relativistic particle.

S7. You can indeed very easily check that

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_{\text{free}}(x,t) = i\hbar\frac{\partial}{\partial t}\psi_{\text{free}}(x,t)\ ,$$

with $\psi_{\text{free}}(x,t) = e^{\frac{i(px-Et)}{\hbar}}$. What you find is $(E - p^2/(2m))\psi_{\text{free}}(x,t) = 0$, which is indeed satisfied because for a free non-relativistic particle $E(p) = p^2/(2m)$.

A8. Define what are the stationary states for the Schrödinger equation.

S8. These are the solutions to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

where E is a parameter with the dimensions of an energy. The importance of stationary states is that if $\psi(x)$ is a stationary state, we can write at once a solution to the time-dependent Schrödinger equation, as

$$\psi(x,t) \;=\; e^{-\frac{iEt}{\hbar}}\psi(x)\ .$$

This represents a state with a definite energy, equal to E. Typically, the energies are discrete and we call E_n the discrete allowed values for the energies, and $\psi^{(n)}$ the corresponding stationary states which solve the time-independent Schrödinger equation with $E = E_n$, i.e.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^{(n)}(x) = E_n\psi^{(n)}(x)$$

Note that a linear combination of stationary states, i.e. $\sum_n c_n \psi^{(n)}(x)$ is not a stationary state (it does not have a definite energy).

A9. Write down the time evolution of $\sum_n c_n \psi^{(n)}(x)$.

S9. It is given by $\sum_n c_n e^{-\frac{iE_n t}{\hbar}} \psi^{(n)}(x)$.

A10. Prove the very important fact that $\sum_n c_n e^{-\frac{iE_n t}{\hbar}} \psi^{(n)}(x)$ IS a solution to the time-dependent Schrödinger equation.

S10. You do not need a calculation to show this, if you make the following considerations. 1. Each term in the sum $e^{-\frac{iE_n t}{\hbar}} \psi^{(n)}(x)$ is a solution to the time-dependent Schrödinger equation. 2. The Schrödinger equation is linear and homogeneous. Hence, any linear combination of solutions is another solution.

Of course you can also easily check directly that $\sum_n c_n e^{-\frac{iE_n t}{\hbar}} \psi^{(n)}(x)$ satisfies the time-dependent Schrödinger equation (and it is a good exercise to convince yourself that this is indeed the case).

A11. For a particle on a segment (i.e. the potential is zero on the segment and infinity outside), consider a wavefunction of the form $\psi(x) = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} \psi^{(n)}(x)$, which we assume to be normalised. $\psi^{(n)}$ are the (normalised) stationary states, i.e. $\psi^{(n)}(x) = \sqrt{2/L}\sin(\pi n x/L)$, with $n = 1, 2, \ldots$.

a. What condition do the coefficients c_n need to satisfy in order for the wavefunction to be normalised?

b. What is the probability of measuring a certain value E_k of the energy?

c. What is the average value of the energy in this state?

S11.

a. We need $1 = \sum_n |c_n|^2$. You arrive at this result by imposing $1 = \int_0^L dx\, |\psi(x)|^2$ and using $\int_0^L dx\, (\psi^{(n)})^* \psi^{(m)} \;=\; \delta^{mn}$.

b. The probability of measuring E_k among the possible values if $|c_k|^2$.

c. It is

$$\langle E \rangle \;=\; \sum_n |c_n|^2 E_n$$

(this is just the average but weighted with the probabilities!)

A12. Write down the normalisation condition for the wavefunction who can move in the one dimensional interval $a \leq x \leq b$.

S12. $\int_a^b dx\, |\psi|^2 \;=\; 1\;.$

A13. State Heisenberg's uncertainty principle for a particle which can move in one dimension.

S13. The uncertainty for a measurement performed at the same time on the position and the momentum of a particle must satisfy the bound

$$\Delta x \Delta p \geq \hbar/2\;.$$

So if we know x very precisely, then the uncertainty on p, Δp, is very large; and viceversa.

A.14 Find the the Broglie wavelength of an electron of energy E =10 eV. Can one use a non-relativistic approximation?

S14. The mass of the electron is $m_e \sim 0.5$ MeV/c^2 and $E/(m_e c^2) = 10/(0.5 \times 10^6) \ll 1$, hence we can use a non-relativistic approximation. Then

$$\lambda_{\text{de Broglie}} = \frac{h}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\hbar c}{\sqrt{2mEc}} = \frac{2\pi\hbar c}{\sqrt{2mc^2E}} \sim \frac{2\pi \times 197}{\sqrt{2 \times 0.5 \times 10 \times 10^{-6}}}\text{fermi} \sim 3.9110^{-10}\text{m},$$

where we have used $E = p^2/(2m)$ and $\hbar c \sim 197$MeV·fermi (1 fermi $= 10^{-15}$m).