Quantum Physics (PHY-215)

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1 From classical physics to quantum physics

1.1 Brief introduction to the course

- The end of classical physics: **1.** Planck's quantum hypothesis (blackbody radiation, specific heat of gases). **2.** The particle behaviour of light (photoelectric effect; Compton scattering). **3.** The wave behaviour of particles.
- Classical instability of matter.
- Orders of magnitudes of some lengths.
- Particles and waves.

1.2 Review of classical mechanics `[Krane 1]`

- Lecture 1. Frame, velocity, acceleration. The concept of "trajectory" of a particle.
- Newton's second law ($\dot{\mathbf{p}} = \mathbf{F}$), conservation of momentum. Conservation of energy.
- Examples of potentials: gravitational potential, harmonic oscillator.
- The harmonic oscillator in detail. Solution to the equation of motion. Angular frequency. Frequency. Period. Average position of the oscillating particle.
- Expansion around the equilibrium position for an arbitrary one-dimensional potential. The importance of harmonic oscillations.

1.3 Review of relativistic kinematics [Krane 2]

- Lectures 2-3. Lorentz transformations. Transformation of the velocities between frames. The speed of light c is constant.
- Velocity and momentum, $\mathbf{p} = m\gamma\mathbf{v}$ with $\gamma := 1/\sqrt{1 - v^2/c^2}$ and $v = |\mathbf{v}|$. Energy. $E^2/c^2 = p^2 + m^2c^2$.
- Approximation for $v/c \ll 1$. Relativistic approximation, $E \sim |\vec{p}|c$.

1.4 Mini-review of electromagnetism [Krane 3]

- Lecture 4. Maxwell equations in vacuum. Wave solutions. Wave number k and wavelength λ, with $k = 2\pi/\lambda$; frequency ω, period $T = 2\pi/\omega$. [Krane 3.1]

1.5 Blackbody radiation and the quantum hypoyhesis

[Krane 3, Bransden-Joachain 1]

- Lecture 5. Thermal radiation. Emissive power (power emitted per unit area from a blackbody with wavelengths between λ and $\lambda + d\lambda$) $dR = R(\lambda, T)d\lambda$. Total power emitted per unit area $R(T) = \int_0^\infty d\lambda\, R(\lambda, T)$.
- Stefan-Boltzmann's law, $R(T) = \sigma T^4$, where σ is the Stefan-Boltmann constant. Wien displacement law for the position λ_{max} of the maximum of the curve $R(\lambda, T)$, $\lambda_{\text{max}}T = b = \text{constant}$.
- Equipartition of energy and Rayleigh-Jeans derivation of $R(\lambda, T)$ based on classical physics: the ultraviolet catastrophe!
- Planck's quantisation hypothesis: the energy of an oscillator is quantised, $E = n\epsilon_0$ where $n = 0, 1, \ldots$.
- Planck's new formula for $R(\lambda, T)$: $R(\lambda, T) = (2\pi hc^2/\lambda^5) \cdot 1/\left(e^{\frac{hc}{\lambda kT}} - 1\right)$.
- Using Wien's displacement law, derivation of ϵ_0: $E = hc/\lambda$, where h is Planck's constant. Alternatively: $E = \hbar\omega = h\nu$ where $\hbar := h/(2\pi)$.
- Lecture 6. Derivation of Wien's displacement constant from Planck's formula for the spectrum of the radiation.

- No ultraviolet catastrophe: derivation of Stefan's law from Planck's formula for the spectrum of the radiation.

- Rayleigh-Jeans formula as a limit of Planck's formula for large wavelentghs.

1.6 A related failure of classical mechanics: specific heats of gases

- Lecture 7. For a gas, the quantity of heat ΔQ needed to raise its temperature by ΔT is $\Delta Q = mc\Delta T$ where m is the mass of the gas, and c is the specific heat of the gas. Mean kinetic energy: $\langle E \rangle = (3/2)K_B T$ for one particle, where $k_B \sim 1.38 \times 10^{-23}$J/K is Boltzmann's constant. Equipartition of energy: each degree of freedom contributes $1/2 k_B T$ to the mean energy.

- Specific heat for a mole of gas: $c_V^{\text{1 mole}} = (3/2)N_A k := (3/2)R$ where $N_A = 6.02 \times 10^{23}$ is Avogadro's number and $R = 8.31\text{J}/(\text{mole}\,\text{K})$.

- Failure of our prediction based on classical physics for the specific heat of biatomic gases, e.g. Iodine I_2. Further failure: specific heats depend on temperature!

1.7 The particle behaviour of light

- Lecture 8-9 Photoelectric effect. Plot of the intensity of the current as a function of the potential V_0. Stopping potential, maximum kinetic energy of the emitted electrons. Einstein's explanation of the photoelectric effect. $(1/2)m_e v_{\max}^2 = h\nu - W$, where ν is the frequency of the light and W is the work function of the metal. Millikan's measure of h/e: from the slope of the plot of $V_0 = (h/e)\nu - W/e$.

- Lecture 10. Compton effect. Derivation of the difference $\lambda' - \lambda = h/(mc)\,(1 - \cos\theta)$, where λ and λ' are the wavelengths of the incident and outgoing light, and θ is the angle between the direction of the incident and the outgoing light (photon).

1.8 The wave behaviour of light and matter

- Lectures 11-12. Review of interference phenomena of light. Interference from a double slit (Young's experiment). Diffraction from a single slit of size a. We can observe diffraction when $\lambda \sim a$, i.e. when the wavelength is of the same order of magnitude of the size of the slit. If $\lambda \ll a$, we are in the realm of geometric optics and we can ignore diffraction. In other words: $\lambda \to 0$: no diffraction. Classical limit.

- Lecture 13. Young's experiment redone: diffraction and interference between two slits in the approximation $d \ll a$ where d=distance of the two slits and a is the size of one slit. Derivation of the interference + diffraction figure. Distance between the fringes.

- Lecture 14. de Broglie's waves. $\lambda_{\text{de Broglie}} := h/p$ where h is Planck's constant and p is the momentum of the particle. Examples of de Broglie wavelengths for different objects – microscopic ones (such as electrons etc) as well as macroscopic. The limit $h \to 0$ is the classical limit.

1.9 Wave-particle duality

- Lecture 15 (and 16). An experiments with bullets. An experiment with water waves. An experiment with electrons.

- Lecture 17. The interference pattern is destroyed whenever we try to observe which slit the electron has passed through. When we do not observe which slit the electron has passed through, we cannot say that the electron pass either through one slit or the other!

- Heisenberg's uncertainty principle. First example: electron going through a single slit. Loss of the concept of "trajectory" of a particle.

- What can we observe? The wavefunction. Introducing the superposition principle using the double-slit experiment.

2 Quantum mechanics

2.1 The wavefunction

- Born's probabilistic interpretation of the wavefunction: $|\psi(\mathbf{x},t)|^2 d^3x$ is the probability of finding the particle in an infinitesimal volume $d^3x := dx\,dy\,dz$ around the point $\mathbf{x}$. Normalisation of the wavefunction: $\int_V d^3x\,|\psi(\mathbf{x},t)|^2 = 1$, where V is the volume where the system is contained. Square-integrable functions.

- Lecture 18. Statement of the superposition principle. If ψ_1 and ψ_2 are two wavefunctions describing the same system, then any linear combination $\psi = c_1\psi_1 + c_2\psi_2$ will also be a possible wavefunction, with c_1 and c_2 being constant (complex) numbers.

- Irrelevance of the overall phase of the wavefunction. Relevance of the relative phase between ψ_1 and ψ_2 in $\psi = c_1\psi_1+c_2\psi_2$: if $\psi_1 := e^{i\alpha_1}|\psi_1|$ and $\psi_2 := e^{i\alpha_2}|\psi_2|$, then $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2|\cos(\alpha_1 - \alpha_2)$.

2.2 Examples of wavefunctions

- Plane waves as solutions for the free particle. Extending the relations valid for photons $E = \hbar\omega$ and $p = \hbar k$, with $\omega(k) = cp$ and $E(p) = cp$ to other particles using de Broglie ideas: we still write $E = \hbar\omega$ and $p = \hbar k$ but now $\omega = \omega(k)$ and $E = E(p)$ will be functions of k and p to be specified later. From $\mathbf{E} = \mathbf{E}_0 \exp[i(kx - \omega t)]$ to $\psi(x,t) = A\exp[i(px - Et)/\hbar]$ (plane waves). Three-dimensional generalisation: $\psi(\mathbf{x},t) = A\exp[i(\mathbf{p}\cdot\mathbf{x} - E(p)t)/\hbar]$.

- The momentum and energy operators for the free particle (plane wave): $-i\hbar\frac{\partial}{\partial x}\psi(x,t) = p_x\psi(x,t)$ and $i\hbar\frac{\partial}{\partial t}\psi(x,t) = E\psi(x,t)$ where the plane-wave solution is $\psi(x,t) = A\exp[i(p_x x - Et)/\hbar]$. Three-dimensional generalisation: $-i\hbar\nabla\psi(\mathbf{x},t) = \hat{\mathbf{p}}\psi(x,t)$.

Reading week (week 7)

- Week 8, Lectures 19-21. Revision of plane wave solutions. Revision of the momentum operator and energy operator. In three dimensions: $\hat{\mathbf{p}} = -i\hbar\nabla$.

- Example of solution of a typical problem: normalisation of the wavefunction. Expectation value of x. Most probable value of x.

- A plane wave is not normalisable. Relative probability. Wavepackets. Phase velocity of a plane wave. Group velocity of a wave packet. The group velocity is equal to the particle velocity. Study of a generic wavepacket (but peaked around a particular momentum). Representation as Fourier transforms. The gaussian wave-packet. Explicit calculation of the Fourier transform of a gaussian wavepacket. Time evolution of the spread of a wavepacket. Examples (microscopic and macroscopic objects).

3 The Schrödinger equation [Krane, Bransden-Joachain]

- Week 9, Lecture 22. Time-dependent Schrödinger equation for a free particle. Generalisation to the case of a particle in a potential $V(x)$ (in one dimension): $i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t)$.

- Properties of the Schrödinger equation: **1.** linearity and homogeneity: the mathematical implementation of the superposition principle. **2.** The equation is first order in the time derivatives – it is enough to know the wavefunction at the time $t = 0$ in order to know it at all times $t > 0$.

- The time-independent Schrödinger equation. Separation of variables. Energy eigenfunctions and energy eigenvalues. $\hat{H}\psi = E\psi$ (the energy eigenfunctions are often called "stationary states").

- Week 10, Lecture 23-24. Particle on a line segment $0 \leq x \leq L$, i.e. $V(x) = 0$ for $0 \leq x \leq L$ and $V = \infty$ for $x < 0$ and $x > L$. Boundary conditions: the wavefunction must vanish at the endpoint of the segment. Solutions for the energy eigenvalues: $E_n = [\hbar^2\pi^2/(2mL^2)]n^2$, where $n = 1, 2, \ldots$. Solution for the eigenfunctions: $\psi^{(n)}(x) = \sqrt{2/L}\sin(\pi n x/L)$. Overlap of energy eigenfunctions: $\int_0^L dx\,(\psi^{(n)}(x))^*\psi^{(m)}(x) = \delta_{nm}$.

- Expectation value of x and of p for different wavefunctions. In particular, if $\psi = \psi^{(n)}$, then $\langle x\rangle_{\psi^{(n)}} = L/2$ and $\langle p\rangle_{\psi^{(n)}} = 0$.

- Calculation of $\Delta x := (\langle x^2\rangle - \langle x\rangle^2)^{1/2}$ and $\Delta p := (\langle p^2\rangle - \langle p\rangle^2)^{1/2}$ and check of Heisenberg's uncertainty principle $\Delta x\Delta p \geq \hbar/2$.

- Collapse of the wavefunction. (New) normalisation of the wavefunction after the measurement.

- Wavefunctions which are linear combinations of energy eigenfunctions and their time evolution. Probability of measuring a certain value of the energy: if the system is in a state described by the wavefunction $\psi = \sum_n c_n\psi^{(n)}$ where $H\psi^{(n)} = E_n\psi^{(n)}$, and the $\psi^{(n)}$ are normalised, then the expectation value (or average value) of the energy is equal to $\langle E\rangle = \sum_n |c_n|^2 E_n$ (used in many problems!).

- Lecture 25. Solution of many typical problems (see the homework assignments and the summary questions you can find on the website).

4 Momentum in quantum mechanics

- Lectures 26-27. Fourier transforms. Dirac delta function. Wavefunction in momentum space $\phi(p)$ and its normalisation. Expectation value of the momentum: $\langle p \rangle = \int_{-\infty}^{+\infty} p\,|\phi(p)|^2 = \int_{-\infty}^{+\infty} \psi^*(x)(-i\,\hbar\,d/dx)\psi(x)$, and hence the momentum p_x is represented by $-i\,\hbar\,d/dx$.

- Lecture 28. One more motivation for the identification $p_x \to -i\,\hbar\,d/dx$: Ehrenfest's theorem: using the time-dependent Schödinger equation we computed $d/dt\,(m\langle x \rangle)$ and we found that $d/dt\,(m\langle x \rangle) = \int_{-\infty}^{+\infty} \psi^*(x)(-i\,\hbar\,d/dx)\psi(x)$.

5 Bohr's model of the hydrogen atom

- Energy levels (stationary states). Frequencies of the radiation emitted/absorbed in the transition between different energy levels.

6 Other applications

- Particle in a three-dimensional box. Solution for the energy levels and the stationary states.

Books

The main textbook for this course is:

1. K. Krane, *Modern Physics* (3$^{\mathrm{rd}}$ edition), John Wiley & Sons.

A more advanced, very nice book, which is adopted in subsequent modules on Quantum Mechanics is:

2. B. H. Bransden and C. J. Joachain, *Quantum Mechanics*, (2$^{\mathrm{nd}}$ edition). Chapters 1, 2, 3 (and part of 4) are very useful also for this course.

Finally, a book which has inspired generations of physicists is:

3. R. P. Feynman, *The Feynman Lectures on Physics, volume III: Quantum Mechanics*. Chapters 1, 2 are a strongly recommended reading.