2) ϕ - Theory (Simpler Toy Example)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

Quartic Interaction (POTENTIAL)

THE INTERACTION PICTURE AND DYSON'S FORMULA

FIRST: BACK TO QM

$$i\frac{\partial}{\partial t}|\psi\rangle_S = H|\psi_S\rangle$$

"Schrödinger Picture"

H independent of t, Operators indep. of t O_S

STATE $|\psi_S\rangle$ depends on t

IN CONTRAST IN THE "HEISENBERG PICTURE"

$$O_H(t) = e^{iHt} O_S e^{-iHt}$$

: Operators depend on t!

$$|\psi\rangle_H = e^{iHt}|\psi\rangle_S$$

STATE $|\psi\rangle_H$ independent on t!

- "INTERACTION PICTURE" IS A HYBRID OF BOTH

SPLIT HAMILTONIAN: $H = H_0 + H_{int}$

H_0: FOR EXAMPLE FREE FIELD HAMILTONIAN

H_{int}: INTERACTION HAMILTONIAN

H_0 USUALLY SOLUBLE

DEFINE $\quad |\psi\rangle_I = e^{iH_0 t} |\psi\rangle_S$

both depend on t

$$O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t}$$

- INTERACTION HAMILTONIAN IN INTERACTION PICTURE $\equiv H_I = (H_{int})_I = e^{iH_0 t} H_{int} e^{-iH_0 t}$

- SCHRÖDINGER EQN. FOR $|\psi\rangle_I$

$$i \frac{\partial}{\partial t} |\psi\rangle_S = H_S |\psi\rangle_S \Rightarrow$$

$$i \frac{\partial}{\partial t}\left(e^{-iH_0 t} |\psi\rangle_I \right) = (H_0 + H_{int})_S \, e^{-iH_0 t} |\psi\rangle_I$$

$$\Rightarrow e^{-iH_0 t}\left(\cancel{H_0 |\psi\rangle_I} + i \frac{\partial}{\partial t} |\psi\rangle_I \right) =$$

$$= \cancel{H_0 e^{-iH_0 t} |\psi\rangle_I} + H_{int} e^{-iH_0 t} |\psi\rangle_I$$

$$\Rightarrow i \frac{\partial}{\partial t} |\psi\rangle_I = \underbrace{e^{iH_0 t} H_{int} e^{-iH_0 t}}_{= H_I} |\psi\rangle_I$$

$$\Rightarrow \boxed{i \frac{\partial}{\partial t} |\psi\rangle_I = H_I |\psi\rangle_I}$$

DYSON'S FORMULA

NOW WE WANT TO SOLVE THIS EQUATION!

WE MAKE THE ANSATZ

$$|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I$$

UNITARY TIME EVOLUTION OPERATOR. IT EVOLVES A STATE KNOWN AT THE TIME t_0 TO THE TIME t.

INSERT INTO THE SCHRÖDINGER EQN. ON PAGE 39

$$\Rightarrow i\frac{\partial}{\partial t}\left(U(t,t_0)|\psi(t_0)\rangle_I\right)$$

$$= i\left(\frac{\partial}{\partial t}U(t,t_0)\right)|\psi(t_0)\rangle_I = H_I\,U(t,t_0)\,|\psi(t_0)\rangle_I$$

$$\Rightarrow \boxed{i\frac{\partial}{\partial t}U(t,t_0) = H_I\,U(t,t_0)} \quad (*)$$

DUE TO OPERATOR ORDERING ISSUES THE SOLUTION IS NOT SIMPLY $U(t,t_0) = e^{-i\int_{t_0}^{t} dt' H_I(t')}$

- INTEGRAL EQUATION FOR U

INTEGRATE $(*)$

$$\Rightarrow U(t,t_0) = -i\int_{t_0}^{t} dt'\, H_I(t')\,U(t',t_0) + \underbrace{C(t_0)}_{\text{CONST. OF INTEGR.}}$$

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WE REQUIRE $U(t_0, t_0) = \mathbb{1}$

$\Rightarrow C(t_0) = \mathbb{1}$, HENCE

$$\boxed{U(t, t_0) = \mathbb{1} - i \int_{t_0}^{t} dt' \, H_I(t') \, U(t', t_0)}$$

THIS CAN BE SOLVED EASILY BY ITERATION

0^{TH} ORDER: $U^{(0)}(t, t_0) = \mathbb{1}$

1^{ST} ORDER: $U^{(1)}(t, t_0) = \left[\mathbb{1} - i \int_{t_0}^{t} dt_1 \, H_I(t_1) \, \mathbb{1}\right]$ $\quad t_0 \leq t_1 \leq t$

2^{nd} ORDER: $U^{(2)}(t, t_0) = \mathbb{1} - i \int_{t_0}^{t} dt_1 \, H_I(t_1) \left[\mathbb{1} - i \int_{t_0}^{t_1} dt_2 \, H_I(t_2)\right]$

$$= \mathbb{1} - i \int_{t_0}^{t} dt_1 \, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2)$$

$t_0 \leq t_2 \leq t_1 \leq t$

$$U(t, t_0) = \text{-----} + (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, .. \int_{t_0}^{t_{n-1}} dt_n \, H_I(t_1) \ldots H_I(t_n)$$

$t_0 \leq t_n \leq t_{n-1} \leq .. \leq t_1 \leq t$

(42) USING THE TIME ORDERING OPERATOR T

$$T\, H_I(t_1)\, H_I(t_2) = \Theta(t_1 - t_2)\, H_I(t_1)\, H_I(t_2) + \cancel{\Theta(t_2-t_1)}$$

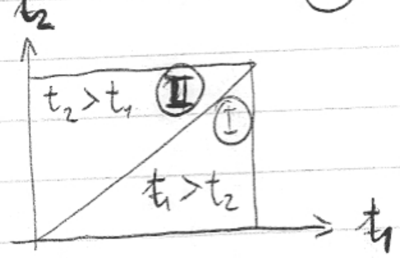
$$\Theta(t_2 - t_1) \times H_I(t_2)\, H_I(t_1)$$

WRITE $\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2)$

$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(H_I(t_1)\, H_I(t_2)\right)$$

<u>PROOF</u> $\frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(H_I(t_1)\, H_I(t_2)\right)$

$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \Big[\Theta(t_1 - t_2)\, \underbrace{H_I(t_1) H_I(t_2)}_{\mathrm{I}} + \Theta(t_2 - t_1)\, \underbrace{H_I(t_2)\, H_I(t_1)}_{\mathrm{II}}\Big]$$



$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \underbrace{\frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \Theta(t_2 - t_1)\, H_I(t_2)\, H_I(t_1)}_{\text{RELABEL } t_1 \leftrightarrow t_2}$$

$\to$ GET SAME INTEGRAL!

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2)$$

THIS EASILY GENERALISES TO n INTEGRATIONS

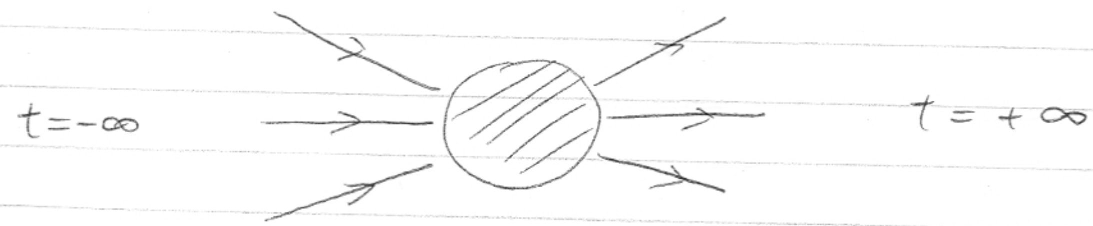
$$\Rightarrow U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n \, T\left(H_I(t_1) \dots H_I(t_n)\right)$$

$$= T \exp\left(-i \int_{t_0}^{t} H_I(t')\,dt'\right)$$

DYSON'S FORMULA!

- THIS EXPRESSION IS RATHER FORMAL; WE USUALLY EXPAND THE EXPONENTIAL UP TO A CERTAIN ORDER UNDER THE ASSUMPTION THAT $H_I(t)$ CAN BE TREATED AS A SMALL PERTURBATION!

SCATTERING AND THE S-MATRIX



INITIAL STATE (FREE PARTICLES) $|i\rangle$

FINAL STATE (FREE PARTICLES) $|f\rangle$

- IN A SCATTERING PROCESS WE TAKE THE INITIAL STATE $|i\rangle$ AT $t=-\infty$ AND THE FINAL STATE AT $t=+\infty$ TO BE EIGEN-STATES OF THE FREE HAMILTONIAN. (THEY CAN BE BUILT BY ACTING WITH CREATION OPERATORS ON THE VACUUM E.G $a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_1)\,|0\rangle$ GIVES A TWO-PARTICLE IN STATE)

- NOW WE CAN EVOLVE THE IN STATE $|i\rangle$ AT $t=-\infty$, USING THE TIME EVOLUTION OPERATOR U, TO $t=+\infty$. & CALL THAT STATE $|\psi\rangle$ THEN,

$$|\psi\rangle = \underbrace{U(+\infty, -\infty)}|i\rangle$$

THIS OPERATOR IS CALLED THE S-MATRIX (S FOR SCATTERING)

- THE PROBALITY AMPLITUDE TO FIND THE INITIAL STATE $|i\rangle$ AFTER SCATTERING IN THE FINAL STATE $|f\rangle$ IS GIVEN BY

$$\boxed{\langle f|\psi\rangle = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} \langle f|U(t,t_0)|i\rangle = \langle f|S|i\rangle}$$

$\langle f|S|i\rangle$ ARE CALLED S-MATRIX ELEMENTS; THERE GENERAL STRUCTURE IS

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\, T_{fi}$$

δ_{fi}: THIS CORRESPONDS TO THE CASE THAT NO SCATTERING TAKES PLACE

$\delta^{(4)}(P_f - P_i)$: MOMENTUM CONSERVATION

T_{fi}: "SCATTERING AMPLITUDE"