

## Lecture 2 (i.e. week 2)

### Pressure as consequence of Newton's second law

[Chap 18.3]  
(pp. 694 ff.)

Last week we saw a conceptually simple but very successful model for what a gas is - let's consider a container of volume

$V$  with some quantity of a gas; we suppose

- 1) - The gas is made out of a very large number of identical objects (molecules/atoms) [4 grams of  $\text{He}_4 = 1 \text{ mole} = N_A = 6,02 \times 10^{23}$  atoms]
- 2) - The size of these molecules is very small in comparison to the average distance between the particles (i.e. we can treat the particle as free and ignore interactions among them).
- 3) - We can describe the dynamics of these molecules just by using Newton's laws (!)
- [4) - The collisions of the molecules against the walls of the container are perfectly elastic ]

From 1-3 + the hypothesis that the temperature is  $[R] 1.12$  nothing else than average energy stored in the molecules ( $\langle \epsilon_{\text{kin}} \rangle = \frac{3}{2} kT$ ) we got a few striking predictions for the heat capacity  $[R] 1.12$

1) - The heat capacity of all mono-atomic gas is the same!

2) -  $C_V$ 's equal to  $\frac{3}{2} R = 12.47 \frac{\text{J}}{\text{K mole}}$

We can generalize these ideas to describe also bi-atomic gases

In this case we're less successful. In this case we should treat the molecules as harmonic oscillator that can both vibrate and move rigidly in space. The result for the heat capacity predicted by our model is now

$$C_v^{\text{biatomic}} = \frac{7}{2} R \approx 29.1 \frac{\text{J}}{\text{K mole}}$$

which is reasonably good in some cases ( $I_2$ ), but completely off for most gases. Here we should postulate that sometimes the molecules become rigid and it is not possible to store energy in the vibrational mode.  $[R]^{1.15}$

This is something that we can not derive from Newton's laws; it's a genuine quantum mechanical effect

⇒ On the other hand I want to show you that the other input we used (namely  $\langle \text{kin} \rangle = \frac{3}{2} kT$ ) can be derived from Newton's laws. Let me focus on a mono-atomic ideal gas. We know that

$$PV = \overset{\# \text{ of moles}}{n} R T = \overset{\# \text{ of atoms}}{N} k T$$

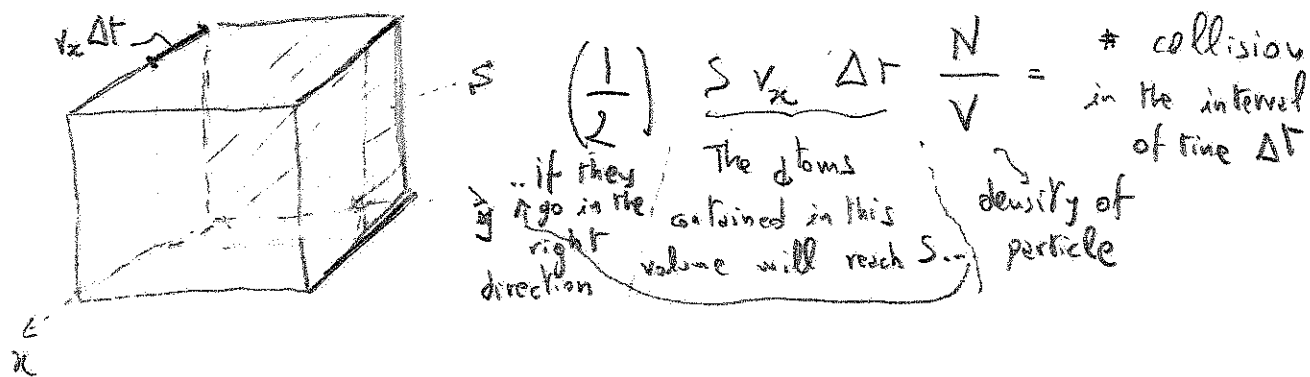
Just from experiment (i.e. take the same amount of gas first at equilibrium at pressure  $P_1$  in the volume  $V_1$  at the temperature  $T_1$  and then change all the quantities to  $P_2$ ,  $V_2$  and  $T_2$ ); we find that

$$\left. \frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2} \right)$$

But what is pressure in our microscopical model?

It is simply due to the many collisions the atoms have against the wall: each collision tries to push the wall away and the sum of all these collisions (which happen continuously) is perceived as a pressure. But if things are like this we can compute this pressure from microscopical data.

1<sup>st</sup> We need to know how many collision we have on a wall each second. Let  $S$  be the surface of this wall



After the collision each atom moves in the opposite direction with the same speed [hyp. 4]. This means that the wall exerted on the atom a force (to change its momentum) and by the 3<sup>rd</sup> law the atom exerted on the wall a force of the same magnitude and opposite direction



The total force felt by the wall in the interval  $\Delta t$  is equal to the total change of momentum of the atoms that collide against the wall in the same interval (it's  $F = \frac{dp}{dt} \Leftrightarrow$

$$F dt = dp \text{ applied for each collision}$$

Total Force felt

$$F \Delta t = \text{by the wall in the interval of time } \Delta t = \left[ \frac{1}{2} \int v_x \Delta t \frac{N}{V} \right] (2 m v_x)$$

# collisions

change of  $p$  in each collision

$$\frac{F}{S} = \frac{N}{V} m v_x^2$$

Up to now we worked as all atoms had the same speed along the axis  $x$ . Of course this is not true, but we can easily extend our proof by thinking that we computed the contribution to the force exerted on the wall by those atoms that do have speed  $v_x$ . Then we apply the same argument (and so the same formula!) to the atoms having speed  $v_x'$ . Finally we define the pressure as the average force / unit of surface and get

$$P = \left\langle \frac{F}{S} \right\rangle = \frac{N}{V} m \langle v_x^2 \rangle$$

As we said in [R]<sup>11.23</sup>  $\langle v_x^2 \rangle = \frac{1}{3} \langle v^2 \rangle$ , so

$$P V = N \frac{1}{3} m \langle v_x^2 \rangle = \frac{2}{3} N \langle K_{in} \rangle$$

By comparing this result with the experimental formula

$$P V = N K T \quad \text{we get}$$

$$\langle K_{in} \rangle = \frac{3}{2} K T \quad \text{as promised!}$$

Boltzmann constant  $\Rightarrow k \approx 1.38 \cdot 10^{-23} \text{ J/K}$  or  $8.62 \cdot 10^{-5} \text{ eV/K}$

So far we considered a gas at equilibrium meaning that its pressure ( $P$ ), its temperature  $T$  and its molecule density

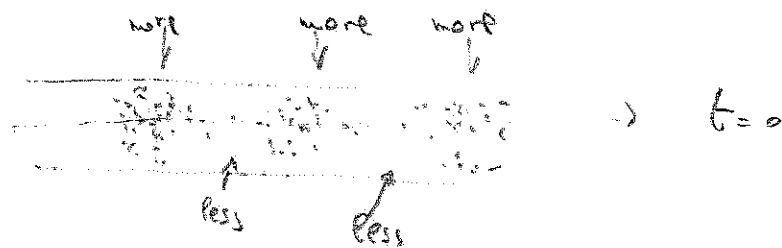
$\frac{N}{V} = \rho$  is the same every-where. Of course there are

many interesting phenomena where this is not the case ...  
for instance "the sound" - What is the sound? Can we explain it in terms of Newton's law applied to atoms/molecules?

Yes!

Basically, the phenomenon of sounds involves 3 features

1°) For a certain reason there is a change of density in the gas  
( $\Rightarrow$  the atoms/molecules pile up in certain places and are more rare in others - consider a tube of gas for simplicity)



The change of density corresponds to a change of the pressure

$$P = \frac{N m_2}{V} \left( \frac{kT}{m_2} \right) \Rightarrow P = B_m \rho \quad \left[ \text{sig. } 418 \right]$$

(In general see 47.2 - 47.4 of Feynman)  $\left[ B_m = \frac{B_{more}}{\rho_0} \right]$

2°) Pressure inequalities generate gas motion  
(by Newton's II law - Pressure is a Force/unit area)

3°) The gas moves and changes the density. So we're back to point 1!! We expect some kind of oscillatory behaviour then...

Each of these 3 steps can be put in formulae and (by using Newton's II law!) we get an equation telling us how the pressure (for instance) varies in space and in time

The solutions of this equation describe mathematically

the situation depicted in the fig. of pag. 2.5. Let us

focus on the pressure at the position  $x=0$  and actually

consider  $P_e(x=0) = P(x=0) - P_0(x=0)$  ( $P_0$  is the standard pressure, the one we would have at the equilibrium, without the sound,  $1 \text{ atm} \approx 1.0133 \cdot 10^5 \frac{\text{New}}{\text{m}^2}$  for normal air).

$$P_e(x=0) = A \cos(\omega t + \phi) \quad \text{something we've already seen!!}$$

What happens if I change point? The pressure will be different (lower in our figure) but then it will change with the same oscillatory behaviour! So we can describe the variation of  $P_e$  in space by saying that the phase  $\phi$  varies with  $x$  (and so get

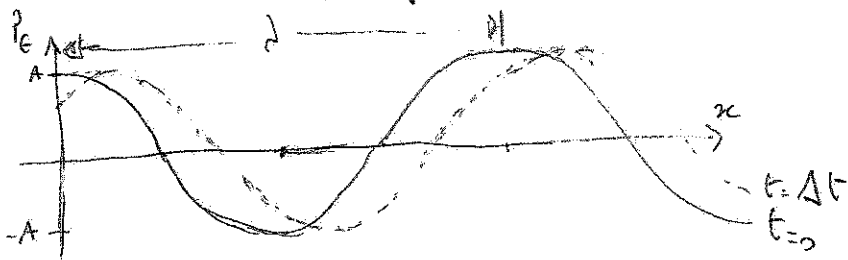
"for free" that the oscillation in time of all points is the same). Let us pose  $\phi = -kx$

$$P_e(x, t) = A \cos(\omega t - kx) \quad [ \phi(k=0) = 0 \text{ as in the picture } ]$$

In fact this is the solution of the equation for  $P$  we could derive from points 1/2/3 if we put them in mathematical language (see Feynman 47.3)

$$\boxed{\frac{\partial^2 P_e}{\partial t^2} = B_m \frac{\partial^2 P_e}{\partial x^2}} \quad \text{wave equation!}$$

with  $B_m = \left(\frac{\omega}{k}\right)^2$  - so let me draw a graph of  $P_e$



• The distance between two maxima  $\lambda = \frac{2\pi}{k}$   $\rightarrow$   $\lambda$  wave-length  
 $\searrow$   $k$  wave number  
 [Just from the Eq above for  $P_e(x, t)$ ]

• At time  $t_0$  there is a maximum at the position  $x=0$ . At time  $\Delta t$  this maximum is at position  $x = \frac{\omega \Delta t}{k}$ . So the velocity of this maximum (i.e. the velocity of the wave) is  $\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} = v_s = \frac{\omega}{k} = \sqrt{B_m}$

$\Rightarrow$  So we connected the speed of sound in the gas to a property of the gas itself (its compressibility) !! 12.7

Notice that the atoms in the gas do not need to move coherently in the direction of the wave (that would be wind, not sound!). What is propagating is not matter, but the disturbance in the pressure... that is energy! The energy carried by a wave  $\sim$  (amplitude of the wave)<sup>2</sup> [chap 15.5]

[This is quite natural: think about the example of the harmonic oscillator] -

• Intensity: in a wave that propagates in 3D, we usually talk about intensity that is the energy carried by the wave per unit area and time  $\frac{dE}{A dt} = I$ . We expect that (in 3D) the intensity falls with  $1/r^2$  ( $r$  being the distance from the source), just by energy conservation.

• The energy transmitted by the wave is different in the various points (F.v, which means that is related to  $\frac{\partial P_c}{\partial x}$ , the difference of pressure being related to the net force)

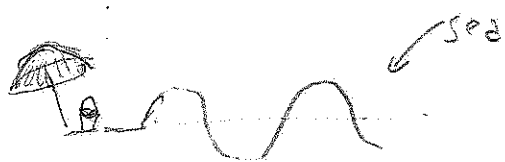
• So it is useful to introduce average intensity (telling us the energy can be transmitted by  $\frac{1}{4}$  wavelength of our wave) and since  $\langle I \rangle = \frac{1}{T} \int_0^T I(t) dt$  with  $I(t) \sim (\text{wave})^2 \sim \sin^2(\omega t - kx)$  we know that the integral will just give a factor of  $\frac{1}{2} \Rightarrow \langle I \rangle \sim \frac{1}{2} A^2$



So we can summarize:

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}} \quad \textcircled{W}$$

where  $\psi$  is the quantity that has the oscillatory behaviour (the stuff the wave is made of). It is pressure for the sound, but it is the water level for sea waves



Other waves in nature?

[Notice a difference between sea-waves and the sound

sound  $\Rightarrow$  longitudinal wave, i.e. the disturbance is in the same direction the wave moves along

sea  $\Rightarrow$  transverse wave, i.e. the disturbance is in the transverse direction with the respect of the wave motion (up/down)

rem: actually real sea-waves contain both types so are a mixture ]

Wave structure.  $\psi = A \cos(\omega t - kx)$

$\omega$  = angular frequency  $k$  = wave number

$\lambda = \frac{2\pi}{k}$  wave-length  $v = \frac{\omega}{k}$  = phase velocity

$\Rightarrow$  SUPERPOSITION PRINCIPLE: if  $\psi_1$  is a solution of  $\textcircled{W}$

[i.e. one wave] and  $\psi_2$  another solution [i.e. another wave]

$\psi_1 + \psi_2$  is still a solution [i.e. an admissible wave]

# Electromagnetic waves

A great discovery of XIX century physics is that light is a wave ... in particular it is an electromagnetic wave (i.e. a fluctuation of the electric and magnetic fields)

$$\vec{E} = (E_0 \sin(kz - \omega t), 0, 0)$$

$$\vec{B} = (0, \frac{E_0}{c} \sin(kz - \omega t), 0)$$

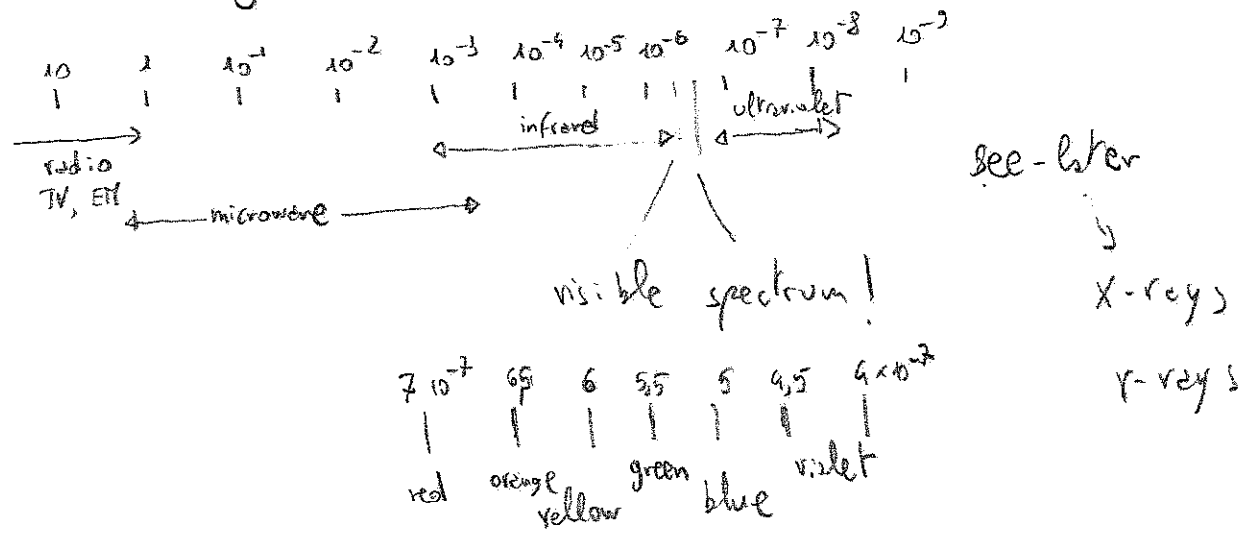
[Notice that Maxwell eqs. require that  $B_y = \frac{E_x}{c}$   
 $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ,  $\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ ]

Notice that both  $\vec{E}$  and  $\vec{B}$  satisfy the wave eqs with  $v = \frac{\omega}{k} = c$

$$\frac{\partial^2 \vec{E}_x}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} \quad c = \frac{\omega}{k} \quad \left[ \begin{array}{l} \text{Implied by Maxwell eqs.} \\ \frac{\partial}{\partial z} [E_0 \sin(kz + \omega t)] = -\frac{\partial}{\partial t} [\frac{E_0}{c} \sin(kz + \omega t)] \end{array} \right]$$

$$\frac{\partial^2 B_y}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 B_y}{\partial t^2}$$

The wavelength of the E.M. wave determines how we perceive it



... but this are just names: we're always talking about the same thing: E.M. waves!!

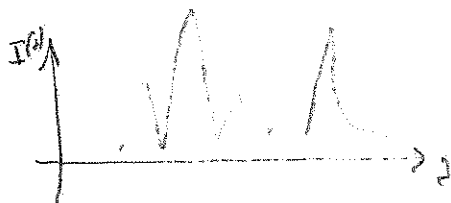
## The Black-body radiation

Solid bodies emit E.M. radiation and the type of E.M. emitted (its wavelength) depends on their temperature (a very useful fact exploited by mosquitoes, unfortunately!). So the total intensity of E.M. waves emitted by a hot body can be split up into the contribution to the intensity due to <sup>radiation of</sup> wavelength  $\lambda$

$I(\lambda)$   $[ I = \int_0^{\infty} I(\lambda) d\lambda ]$ . So is  $I$  has dimensions

$$\frac{\text{Energy}}{\text{sec Area}}, \quad I(\lambda) \sim \frac{\text{Energy}}{\text{sec Vol.}}$$

The spectrum of emitted radiation (intensity) in function of  $\lambda$  is

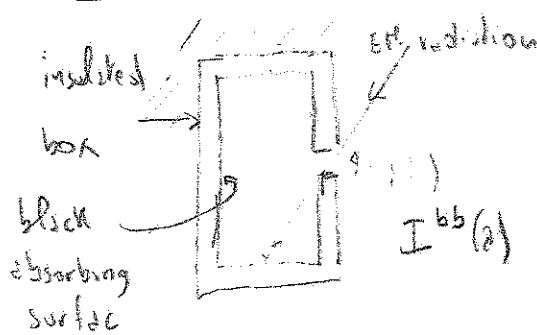


very irregular in general.

At the end six century physicist studied the  $I(\lambda)$  spectrum of various materials, also to see what the best way was to make light (oil-lamps, electricity lamps). This study had a huge theoretical importance in the discovery of Q.M. This is the story. As usual, things became theoretically very interesting when people realized that there is an (idealized) object whose spectrum is much more regular than those of normal materials:

The Black body - What is it? Think about an object at  
 ① equilibrium at temperature  $T$  (no heat exchanged with the environment), ② that doesn't reflect any light but absorbs all E.M. radiation hitting it (this is why it's <sup>called</sup> black)  
 ③ re-emits all the radiation absorbed ... quite a strange and remarkable object! Any examples of such a thing?

### In the laboratory

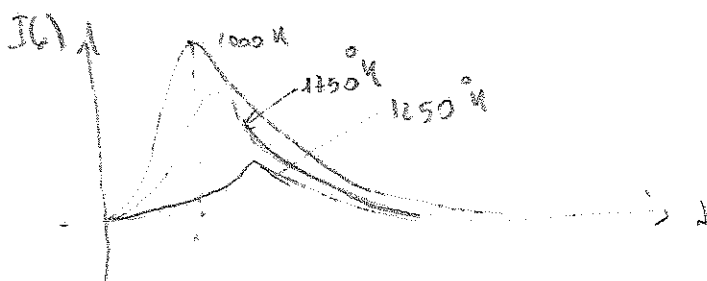


- 1) box heats up a bit
- 2) Thermal radiation in equilibrium inside the box
- 3) some of this radiation (not the original one!) escapes from the hole

### In astronomy

- 1) Stars can be approximately treated as black bodies.
- 2) Black holes: are they exactly blackbodies or is it still an approximation? Nobody really knows...

Anyway from the near blackbodies we can construct in the lab. we can find experimentally a great deal of information



1) very regular shape ... Can we derive the form of the function  $I(\lambda)$  theoretically?

2) Stefan-Boltzmann law (exp.)

$$I = \int I(\lambda) d\lambda = \sigma T^4 \quad \frac{J}{m^2 s K^4} = 5.67 \cdot 10^{-8} \frac{J}{m^2 s K^4}$$

and very general **IMPORTANT**  
 $I(\lambda)$  does not depend on  
 1) shape  
 2) material  
 3) colour  
 of the curve but just on the TEMPERATURE

3) Wien displacement law - Let  $\lambda_m$  be the maximum of the function  $I(\lambda)$  (i.e. the wave length contributing most to the total intensity), then

$$\lambda_m T = \text{constant} = 2.90 \times 10^{-3} \text{ m} \cdot \text{K}$$

Rayleigh tried to explain point 1) by using the standard wave-dynamics for E.M. radiation inside the cavity ... but failed!

This is brief sketch of his approach:

- In the cavity there are E.M. of different frequencies. By the equipartition theorem we expect that the energy store in each frequency depends just on the temperature  $\langle E_i \rangle \sim kT$
- So the intensity  $(I(\lambda) \sim \frac{E}{m^3 s})$  must be proportional to  $kT$

$$I(\lambda) \sim \left( \frac{\dots}{\lambda^2} \right) kT \quad \leftarrow \text{dimension of energies}$$

What can I put here to get the right dimensions?

I can't put anything that has to do with the cavity itself (how it's built etc.) because the spectrum  $I(\lambda)$  is independent of all these details!

- I can just put  $\lambda$  (the wave length) [metre]
- The speed of light  $c$  (a constant everyone agrees on!)
- Nothing else!

Then  $I(\lambda) \sim \frac{c k T}{\lambda^4}$  is the only possibility

If I fix the coefficient to be  $I(\lambda) = \frac{2\pi^5 c k T}{15 \lambda^4}$

then this function fits very well the experimental data for large wave lengths ... but it fails badly for small  $\lambda$ .

• We can not get Wien's law

• The total intensity predicted  $\sim \int_0^{\infty} \frac{d\lambda}{\lambda^4} = \infty$  !!

Problems like this one are usually called UV problems

They deal with very small distances and these infinities usually signal that there is something serious we do not understand about the microscopic structure of our object of study... light in this case -