The wave-nature of particles

So far we focused on the study of light (and other E.M. waves). Planck's idea of quantization of energy works beatifully to explain some experimental facts that otherwise would be puzzling

- Photoelectric effect (treshold frequency ; $\nu_0 = \frac{h}{e}\nu - \frac{\phi}{e}$ ⇐ non-relativistic collision
- Compton scattering ($\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$ ⇐ relativist collision)
- Blackbody radiation (yet to really explain to you how to derive Planck's radiation law $\boxed{R4.4}$, but I'll do it!)

So we <u>might</u> conclude at this point that the concept of particle is the really fundamental one and that light is <u>almost</u> like the sound, a wave describing the density of these photon (why almost? Remember that sound is a longitudinal wave, while light is a transverse wave - with a polarization ... however this is not the real problem). However this simple interpretation is NOT correct!

Let me give you a reason for its failure (more later!)

- We know that something is wrong also about particle mechanics in the microscopical world, because in the study of gases sometimes we get wrong results for their thermodynamics properties (for instance the specific heat of biatomic gases)

From Rutherford scattering experiments it looks like the atom is a small planetary system, with negative charged particles (the electrons) orbitating around positive charged and a heavy nucleus. Why is it stable (contrary to what 2nd Newton's law would predict?).

In fact in 1924 de Broglie put forward (in his PhD-thesis) this revolutionary proposal:

"Particles have also a wave-like nature, exactly as light has a particle-like nature"

Lot of philosophy steemed out from this hypothesis (complementarity principle, etc.), but let us leave this approach to philosophers and write in formulae what de Broglie meant to <u>really</u> understand what's going on.

For photons we have $\boxed{E = h\nu}$; Special relativity $E = pc$ $\Rightarrow$ $\boxed{p = \frac{h\nu}{c} = \frac{h}{\lambda}} = \hbar k$

We have seen that this momentum is real (!) in the Compton scattering

Then de-Broglie proposed that for <u>any</u> particle we have

any particle $\nu_{dB} = \frac{E}{h} = \frac{\gamma m c^2}{h}$ and $\lambda_{dB} = \frac{h}{p} = \frac{h}{\gamma m v}$

Thus we can associate a wave (with a well-specified frequency and wavelength) to any particle!! If $v \ll c$ ($\beta = \frac{v}{c} \ll 1$) then we can use the non-relativistic approximation $\lambda_{dB} \sim \frac{h}{mv}$

What are these de-Broglie waves? At the beginning all this appeared very strange and it became even more strange when this wave-nature of particle was succesfully applied to re-derive an hypothesis Bohr put forward ten years before (1913). Bohr said: We know that both energy and momentum are quantized ((for photons). They should be quantized also for the electrons bound to the nucleus (in any atom). And also angular momentum should by quantized!

<u>Rotherford planetary</u> system for H (the hydrogen atom)

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}(-\vec{n}_r)$$

$$L = m\,v\,r$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{v\,d\vartheta}{dt}(-\vec{n}_r) = v\omega(-\vec{n}_r)$$

For constant angular frequency $r/\omega = v$

$$= \frac{v^2}{r}(-\vec{n}_r)$$

$$\Rightarrow \bar{F} = ma \Rightarrow \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m_e\frac{v^2}{r}$$

$$\Rightarrow v = \frac{1}{4\pi\epsilon_0}\frac{e^2}{(m_e v r)}$$

<u>Bohr hypothesis</u>

$L = m_e v r$ is quantized

$$\boxed{L = n\frac{h}{2\pi}} \quad n = 1, 2 \ldots$$

with this hypothesis v is quantized

$$v_n = \frac{e^2}{2\epsilon_0 n h}$$

and the radi are quantized

$$r_n = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e v_n^2} = \frac{\epsilon_0 n^2 h^2}{\pi m_e e^2}$$

Consequences:

- Atoms are stable (the radius is quantized and can not change continuously).
- We get an estimate for the size of the hydrogen atom (compute $r_1 = 0.529 \; 10^{-10}$ m)
- The energy levels are discrete

Fine structure constant

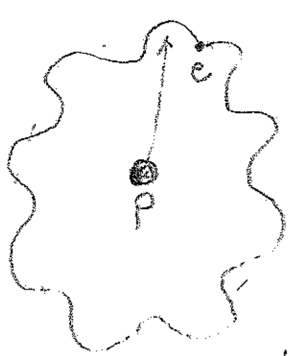
$$\alpha = \frac{e^2}{(4\pi\varepsilon_0)\hbar c} \approx \frac{1}{137}$$

$$E_n = \underbrace{\frac{1}{2} m_e v_n^2}_{K_n} + \left(-\frac{e^2}{4\pi\varepsilon_0 r_n}\right) = -\frac{1}{\epsilon_0^2} \frac{m e^4}{8 n^2 h^2}$$

and thus when the electron move from one energy-level to the other they emit/absorb photons of a fixed energy (frequency).

So Bohr's hypothesis is able to explain many experimental facts... but why is the angular momentum quantized ??

Let's go back to de-Broglie waves: we notice that the allowed orbits for the electron, are exactly those for which the corresponding wave closes nicely! A wave can "fit" into an orbit only if the circumference of the orbit is an integer multiple of its wave-length



$$2\pi r_n = n\lambda \Rightarrow \frac{r_n}{\lambda} = \frac{n}{2\pi}$$

$$\Downarrow \text{ de. Broglie}$$

$$p \cdot r_n = \hbar n \Leftarrow \frac{h}{\lambda} r_n = \frac{h n}{2\pi}$$

$$\Downarrow \text{ non-rel. approx}$$

$$\boxed{L_n = m v_n r_n = \hbar n}$$ which is Bohr's hypothesis!!

SPECTROSCOPY

We have just seen that the energy levels for the electron in a hydrogen atom are discrete and only very particular values of E are allowed. This fits very well the following observation: hydrogen gas seems to emit/absorb light in preference for very particular frequencies/wavelength. Experimentally this was known well before Bohr's model of the atom (and was seen as a puzzle!)

In 1885 a swiss mathematician Johann Balmer realized that some of the wavelengths absorbed/emitted by H have a peculiar property:

$\lambda_{B_4} = 410$ nm $\quad \lambda_{B_3} = 434$ nm $\quad \lambda_{B_2} = 486$ nm $\quad \lambda_{B_1} = 656$ nm

He realized that $\quad \lambda_{B_m} = \text{Const.} \dfrac{(m+2)^2}{(m+2)^2 - 4}$

- From m=1 $\quad \text{Const} = \dfrac{5}{9}\, 656 \text{ nm} \approx 364.4 \text{ nm}$

m=2 $\quad \lambda_{B_2} = 364.4 \dfrac{16}{12} \text{ nm} \approx 486 \text{ nm}$

$\lambda_{B_3} = 364.4 \dfrac{25}{21} \text{ nm} \approx 434 \text{ nm}$

......

$\lambda_{B_5} = 364 \dfrac{49}{45} \approx 397 \text{ nm}$!

Clearly there must be a reason for such a regularity.

In the year after Balmer's observation, more experimental data were obtained, by looking also at the spectral lines outside the visible spectrum. Lyman discovered another regular series

$\lambda_{L_1} = 122$ nm $\lambda_{L_2} = 103$ nm $\lambda_{L_3} = 97.2$

which followed a pattern similar to Balmer's one

$$\lambda_{L_m} = \hat{Const} \frac{(m+1)^2}{(m+1)^2 - 1}$$

Not surprisingly other series were found ... and Rydberg found a unifying formula for all of them $(m > n)$

$$\frac{1}{\lambda_{m,n}} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \Rightarrow \lambda_{m,n} = \frac{1}{R} \frac{m^2 n^2}{m^2 - n^2}$$

where $R = 1.097 \ 10^7 \ m^{-1}$. Then we have that $n=2$ gives Balmer's formula, $n=1$ Lyman's one etc.

The great immediate success of Bohr's theory was to derive Rydberg's formula. The idea is simply that the energy of the photon (light) emitted by an hydrogen atom must be equal to the difference of energy between two different levels (by energy conservation).

Thus if the electron moves from the orbit m to a lower (less energetic) orbit n ($m > n$) then the energy of the photon is given by $E_p = E_m - E_n$. In formulas

$$E_p = \frac{hc}{\lambda} \quad \text{and} \quad E_n = -\frac{m_e e^4}{8\epsilon_0^2 n^2 h^2} \quad \text{Thus}$$

$$\frac{hc}{\lambda_{m,n}} = + \frac{m_e e^4}{8\epsilon_0^2 h^2}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \Rightarrow$$

$$\frac{1}{\lambda_{m,n}} = \frac{m_e e^4}{8\epsilon_0^2 h^3 c}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

$$R = \frac{m_e e^4}{8\epsilon_0^2 h^3 c} = \frac{1}{2}\left(\frac{e^2}{2hc\epsilon_0}\right)^2\left(\frac{m_e c}{h}\right) = \frac{1}{2}\alpha^2\frac{1}{\lambda_c}$$

α fine structure $\alpha \approx \frac{1}{137}$,

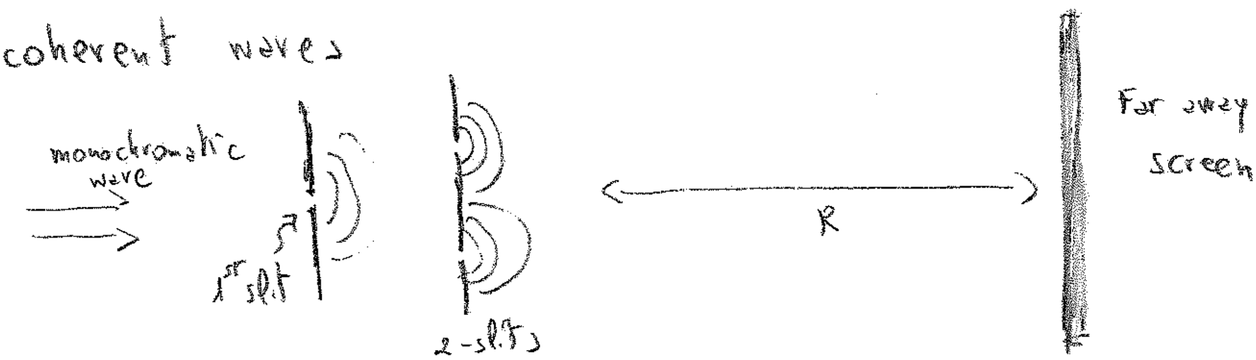
λ_c Compton's wavelength $\lambda_c \approx 2.43\ 10^{-12}$ m

... but are these waves real ??

DIFFRACTION

We know that one distinguishing feature of waves is the phenomenon of interference. Afterall Young used this idea to prove the wave-like nature of light (which we're disputing now!). So we could try and follow the same approach with the de Broglie waves: to see whether there's really a wave associated to an electron with momentum p (with wavelength $\lambda = \frac{h}{p}$) let's repeat Young experiment and see if we get any interference fringe!

At this point we need to rethink the details of Young's experimental apparatus: 1) He used one slit to produce a synchronized wave propagating from a "point"; 2) then he used 2 slits to produce two coherent waves



Now the crucial question is how big should the slits be?

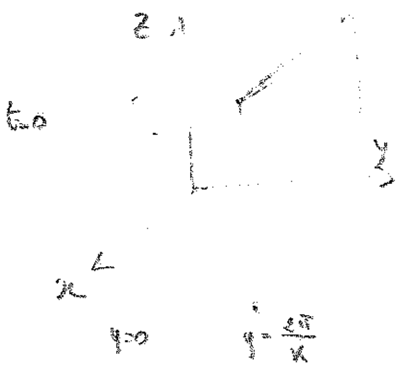
answ: They should be rather small $10^{-6}\,m = 1\mu m$ each to see any interesting effect (bigger slits do not produce interference)

Two questions:

1) why should the slits be so small?

2) How small should they be for seeing interference among electrons?

Let me start from question number ①. To explain this let me introduce what is called Huygens' principle (but it should be more properly called Huygens' observation since it is just a consequence of the wave equation!). Consider a monochromatic wave with plane wave front propagating along the y-direction



$$\psi(y,t) = A \cos(\omega t - k y)$$

two front waves at $y=0$ and $y=\frac{2\pi}{k}$

Huygens: $\psi(y,t)$ can be seen as the superposition of many spherical waves located on the wave-front and propagating radially (neglect the waves going backwards!).

$$\psi(y,t) = \sum \frac{A}{r'} \cos(\omega t - k r')$$

[Notice the coefficient is $\frac{A}{r'}$ because we know that spherical waves have intensities that decay as $1/(r')^2$!].

⇒ Thus when a wave (or better a plane-wave, i.e. a wave with plane wave-front) encounters a slit we can reconstruct

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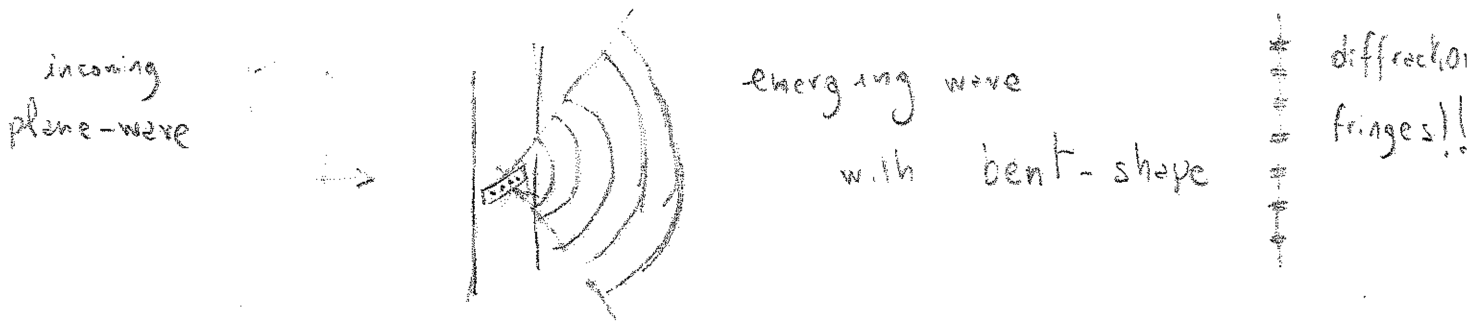
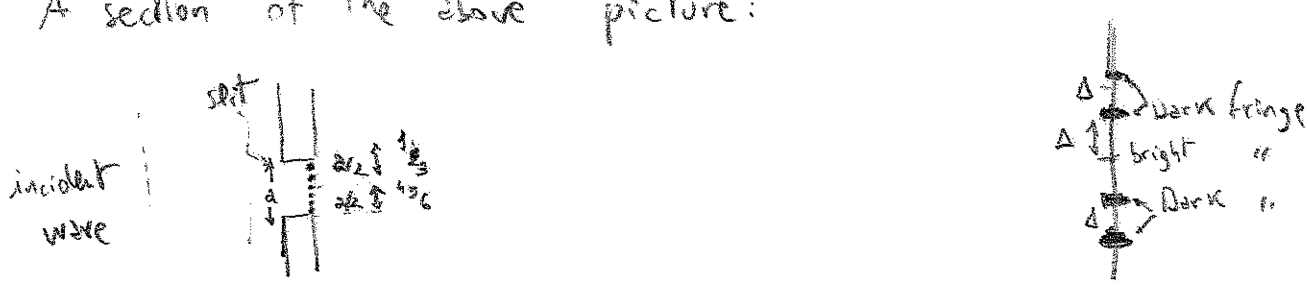
the emerging wave by superimposing the spherical sources that stay in the slit



If we project this emerging wave on a distant screen we see a familiar pattern: dark and bright fringes. We can understand the reason for this phenomenon by applying Huygens' principle and by computing the interference of the (many) point-like sources "in the slit".

A section of the above picture:



The interference among many point-like sources is usually called diffraction (and the name interference is reserved for the case where we have few sources and mainly only two sources.). But this is just a nominalistic issue; in all cases what we have to do is just sum up the contributions of different waves. In order to do this we can use our previous results for the case where we have two sources that we studied for the (idealized) Young experiment

In that case the result for the position of the first dark fringe was $\boxed{\Delta = \frac{\lambda R}{2d}}$ ← (for small angles), where λ is the wavelength and d is the distance of the point line sources.

Now we should sum up all contributions in the figure above: we do it by pairing the $2n$ (Huygens) sources in the following way $(1, n+1)$ $(2, n+2), \dots, (n, 2n)$. The distance between the 2 sources in each pair is $a/2$. So each pair of sources give rise to a destructive interference iff $\Delta = \frac{\lambda R}{a}$ ($d = a/2$ in the formula above). Thus the contribution of the first half of the fringe cancels that of the second half ⇒ thus we get a destructive interference and then a dark band.

Next we can devide the whole slit in 4 parts and ask that the contribution of the first part cancel those of the second and similarly for parts 3 and 4. Then we apply again the formula above, but now with $d = a/4$. Thus the next dark bands are for $\Delta_2 = \frac{2\lambda R}{a}$. And in general (when we devide the slit in 6, 8, ..., 2n parts) we have more dark bands at distances $\Delta_m = \frac{m \lambda R}{a}$ $m = 1, 2 \dots$ from the center.

(REMEMBER: We are in the approx. $\theta \sim \frac{\lambda}{a} \ll 1$)

What about the intensity of the wave emerging from the slit?

Again we can use Huygens' trick; this time let us sum up explicitely the contributions of all the sources. In practice we have to sum up many cos functions whose phase is slightly different:

$$\Delta\phi = k \sin\vartheta \, \Delta l$$

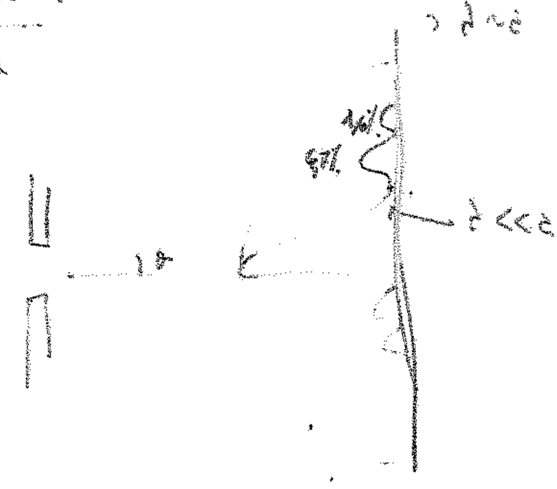
$$\left[\cos(\omega t - kr) + \cos(\omega t - kr + \Delta\phi) + \cos(\omega t - kr + 2\Delta\phi) + \dots\right]$$

$$\mathrm{Re}\left[e^{i(\omega t - kr)} \sum e^{in\Delta\phi}\right] = \mathrm{Re}\left[e^{i(\omega t - kr)} \frac{1 - e^{iN\Delta\phi}}{1 - e^{i\Delta\phi}}\right]$$

$$= \mathrm{Re}\left[e^{i(\omega t - kr + (N-1)\Delta\phi)} \frac{\sin\frac{N\Delta\phi}{2}}{\sin\frac{\Delta\phi}{2}}\right] =$$

$$= N \cos(\omega t - kr + (N-1)\Delta\phi) \frac{\sin\left(\frac{kd\sin\vartheta}{2}\right)}{\frac{kd\sin\vartheta}{\lambda}}$$

$$\langle I \rangle \sim \frac{\sin^2 \frac{\pi d \sin\vartheta}{\lambda}}{\left(\frac{\pi d \sin\vartheta}{\lambda}\right)^2}$$



$\frac{\pi d \sin\vartheta}{\lambda} = \pi$ is the position of the first dark fringe

⟹ $\boxed{\sin\vartheta = \frac{\lambda}{d}}$ ... so to have a big ϑ ⟹ $\lambda \sim d$

↗ if $\lambda \ll d$ ⟹ wave (just like particles) does not bend

↘ if $\lambda \sim d$ ⟹ then the wave bends!