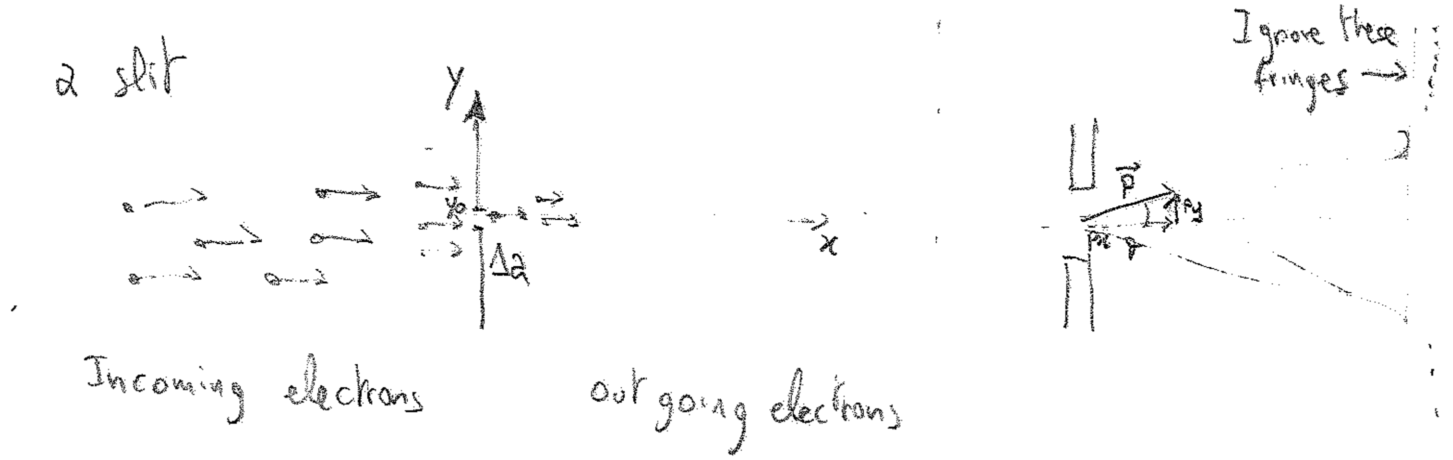
Heisenberg uncertainty principle

We saw that particles (like the electron) displays wave-like features (like the interference in the Davisson-Germer experiment). What would happen is we try to set-up a double-slit experiment for electrons as Young did with light? Setting up such an experiment is very difficult... thought to be impossible (we know the reason: the tipical wavelength of an electron of few tents of eV is $\sim 10^{-10}$ m., 1000 times smaller than visibile light and the 2-slit apparatus should have sizes not too much big than λ!).
But the group of A. Tonomura succeded in 1989 to setup this experiment (Am. Journ Phys 57 (1989) 117).
See pag. 1500 of Young-Freedman or http://www.hqrd.hitachi.co.jp/em/doubleslit.html
Electrons are sent one by one against the "double-slit", they are detected one-by-one, but after a while their distribution forms the interference pattern.
What does that mean? What is the trajectory of the electron?
In order to see this let us see what happens when we try to localize (i.e. specify the position) of an electron.
Conceptually the easiest way to do this is to use



After the slit the beam of electron should be focused in the y directions so we should now their position along the axis y with an uncertainty of $\Delta a/2$ ($y = y_0 \pm \Delta a/2$). But what happens if we want a very precise measure of the y-position (i.e. if we make Δa very small)? We know from Davisson-Germer or Tonomura experiments that when Δa becomes comparable with the de Broglie wave-length of the electron we will see diffraction! That is the beam of electrons after the slit is <u>not</u> focused any more but spreads with an agle ϑ satisfying

$$\boxed{5.9} \quad \sin\vartheta = \frac{\lambda}{a} \xRightarrow{\lambda/a \ll 1} \vartheta \sim \frac{\lambda}{a}$$

(here we will work with small angles for which $\sin\vartheta \sim \tan\vartheta \sim \vartheta$). This means that after the slit the electron <u>may</u> have a velocity (or better a momentum) also along the direction y (either in the positive or in the negative direction). The maximal value of this p_y is (please ignore for the moment the other fringes)

$$\frac{|p_y|}{|p|} = \sin\vartheta \qquad \text{with} \qquad |p| = \sqrt{p_x^2 + p_y^2}$$

So our uncertainty on p_y is bounded as follows

$$|\Delta p_y| \gtrsim \sin\vartheta \, |p| \quad (*)$$

(Note Δp_y can be bigger than $|p| \sin\vartheta$ because there are other smaller fringes at bigger y-distances).

Now let me use de Broglie hypothesis $|\vec{p}| = \frac{h}{\lambda}$ =>

$$\frac{h}{\lambda} = \sqrt{p_x^2 + p_y^2} = |p| \quad \& \quad \sin\vartheta = \frac{\lambda}{d} \quad (1)$$

and the equation telling us how much the beam spreads $\vartheta \sim \frac{\lambda}{d}$ (2)

(1) and (2) => $|p| = \frac{h}{\sin\vartheta \, d}$ and using this in (*)

$\Downarrow$

we get $\quad |\Delta p_y| \gtrsim \frac{h}{\cancel{\sin\vartheta} \, d} \cancel{\sin\vartheta} \gtrsim \frac{h}{d}$

$\Downarrow$ (use $|\Delta y| = d/2$)

$$|\Delta p_y| \, d \gtrsim h \quad \Rightarrow \quad |\Delta p_y| \, |\Delta y| \gtrsim \frac{h}{2}$$

Thus we <u>can not know</u> the momentum (~ velocity) and the position along the same axis at the same time. Heisenberg uncertainty principle. The exact statement is

$$\boxed{|\Delta p_y| \, |\Delta y| \geq \frac{\hbar}{2}}$$

Notice that this is an uncertainty that has <u>never</u> been seen before in physics! It is not something that has to do with our technical limited ability: even if we are able to make a slit very thin (a small) we can not keep the electron beam focused. It is something <u>intrinsic</u> to nature!!

If this is the case, then in the microworld of particles we have to give up a few concepts we are used to

- the idea that particles follow a well-defined trajectory
- the idea that particles can always be distinguished by their position

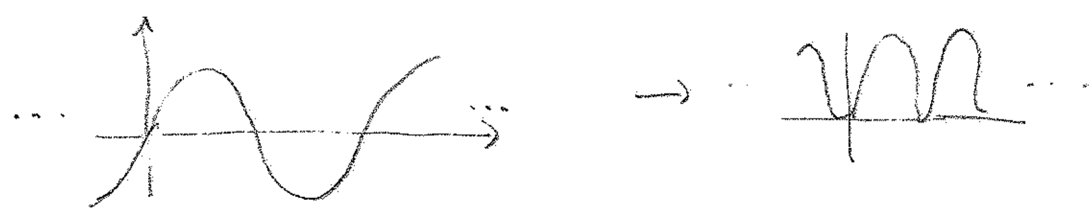
The micro-world <u>is</u> very different from the one we are used to and we need new equations and a new way of interpreting them in order to describe it. Let me start from the interpretation.

<u>1st (sloppy) version of Born interpretation of de-Broglie waves:</u> these waves describe the <u>probability</u> distribution of the particle. Since we can not describe exactly the trajectory, we can just say that the particle is <u>likely</u> to be in a certain region of space and is <u>likely</u> to move in other region in the next instant. The <u>square</u> of the de-Broglie wave describe this probability distribution ($(\psi(x))^2 \Delta x = P(x)$). Why do we need the square? Two arguments

1) $(\psi(x))^2$ is always positive, 2) From Tonomura's experiment we know that $P(x) \cdot N$ looks like the intensity of Young's experiment

which is related to the square of the wave forming the interference pattern.

Thus in agreement with Heisenberg uncertainty principle when we describe a particle (photon or electron) by means of an "infinite" and regular wave like $\sin(\omega t - kx)$

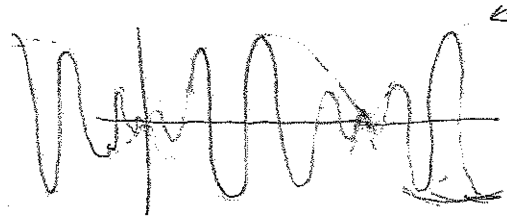


we know exactly its momentum $\left(p = \frac{h}{\lambda} = \hbar k\right)$, but we have no information on it's position since $P(x)$ does not go to zero as $x \to \pm\infty$ (and in the non-sloppy formulation we will see that $P(x)$ will be completely flat!). So how can I describe a particle that is (more or less) localized? By means of wave-packets which are superposition of "infinite" waves with almost the same frequency.

$$\psi_1 = A \sin(\omega t - kx) \qquad \psi_2 = A \sin\left[(\omega + \delta\omega)t - (k + \delta k)x\right]$$

$$\Rightarrow \psi_1 + \psi_2 = 2A \cos\left(\frac{\delta\omega}{2}t - \frac{\delta k}{2}x\right) \sin\left[\left(\omega + \frac{\delta\omega}{2}\right)t - \left(k + \frac{\delta k}{2}\right)x\right]$$

envelopping wave

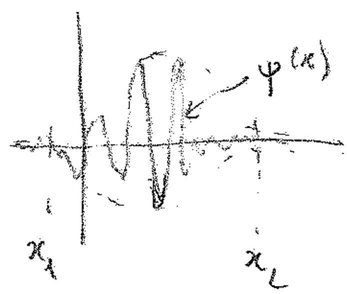


$\delta\omega$ = beat frequency

p. 619

[This is again an interference phenomenon: the wave can get out-of-phase and the in-phase because they travel at difference velocities].

I can superimpose more waves with almost the same frequency ω and wavelength $\lambda = \frac{2\pi}{k}$ to obtain something really localized



$P(x) = (\psi(x))^2$ is really different from zero only in the region between x_1 and x_2

The price we pay is that the momentum of the particle is not exactly defined, because $\psi(x)$ is a superposition of sin-waves with different λ's (which yield different $p = \frac{h}{\lambda}$).

[This explain a mistery I did hide up to now: the velocity of a standard de Broglie wave is bigger than the phase velocity of light!

$$V_p = \lambda_{dB} \cdot \nu_{dB} = \frac{\lambda_{dB}}{h} h \nu_{dB} = \frac{E}{p} \Rightarrow$$

$$V_p^2 = \frac{E^2}{p^2} = \frac{p^2c^2 + m^2c^4}{p^2} = c^2\left(1 + \frac{m^2c^2}{p^2}\right) \Rightarrow V_p = c\sqrt{1 + \frac{m^2c^2}{p^2}}$$

or expressed in terms of the wave data $V_p = c\sqrt{1 + \frac{m^2c^2}{\hbar^2 k^2}}$

when $m \neq 0$ V_p is a function of k $V_p(k)$!

What is the meaning of this? Well, the particle is described by a wave-packet and not by a sin-wave and the velocity of the particle is actually the velocity of the envelopping wave!

This is called group velocity and from the formula in the previous page is $V_g = \frac{\delta \omega}{\delta k} \Rightarrow V_g = \frac{d\omega}{dk}$.

Let me compute explicitly v_g by using $\frac{\omega}{k} = v_p \Rightarrow \omega = v_p k$

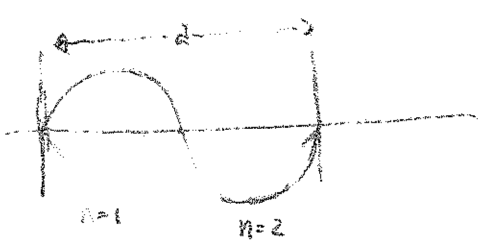
$$v_g = \frac{d}{dk}\left(v_p k\right) = v_p + k\frac{dv_p}{dk}$$

$$= v_p + k\frac{d}{dk}\left(c\sqrt{1+\frac{m^2c^2}{\hbar^2k^2}}\right) = v_p + c\,\frac{\frac{m^2c^2}{\hbar^2k^3}(-2)}{2\sqrt{1+\frac{m^2c^2}{\hbar^2k^2}}}\,k$$

$$= v_p - c^2\frac{\left(\frac{m^2c^2}{\hbar^2k^2}+1-1\right)}{v_p} = v_p - \frac{c^2}{v_p}\left(\frac{v_p^2}{c^2}-1\right) = \frac{c^2}{v_p}$$

$\Rightarrow \quad v_g v_p = c^2$ so if $v_p \geq c$ then $v_g \leq c$!!]

Let me now fix another loose end: consider a particle trapped in a 1-dimensional box of size a. According to the de-Broglie picture of standing waves the particle is described by a wave with $n\lambda = 2a$ (like a guitar string)



So we $\psi(x,t) = A\sin(k_n x)\sin\omega_n t$

where $k_n = \frac{2\pi}{\lambda_n} = \frac{\pi n}{a}$ and

$\omega_n = \frac{E_n}{\hbar} = \frac{\hbar\pi^2n^2}{2ma^2}$

$$E_n = \frac{p^2}{2m} = \frac{\hbar^2k_n^2}{2m} = \frac{\hbar^2\pi^2n^2}{2ma^2}$$

... but at certain times $\left(t_m = \frac{2\pi m}{\omega_n}\right)$ $\psi(x,t_m) = 0$ and

our probability interpretation breaks down (the particle is no-where, has disappeared!).

Solution: the wave describing an elementary particle can be complex and the probability to find the particle under study in the region of space x, $x+dx$ is $\psi^*(x)\,\psi(x) = |\psi(x)|^2$.

From now on we will call this complex wave the "wave function" (the concept has evolved a long way since the initial intuition of de-Broglie and so it deserves a new name!). So

For a particle in a box
$\psi(0,t) = \psi(a,t) = 0$

$$\psi(x,t) = A\sin(k_n x)\left[\cos\omega_n t + i\sin\omega_n t\right] = A\sin k_n x\, e^{i\omega_n t}$$

$$\left(\Rightarrow P(x) = A^2\sin^2 k_n x\; e^{-i\omega_n t}\, e^{i\omega_n t} = A^2\sin^2 k_n x\right)$$

for a particle on a circle
$\psi(0,t) = \psi(2\pi,t)$

$$\psi(x,t) = A\cos(kx-\omega t) + i\sin(kx-\omega t) = Ae^{i(kx-\omega t)}$$

$$\left(\Rightarrow P(x) = A^2 \text{ for any } x \text{ and } t!\right)$$

So how should I fix the size of these waves (the coefficient A)? A natural request is to ask that the probabilities add up to 1 (the particle must be somewhere!)

$$\int P(x)\,dx = \int \psi^*(x)\,\psi(x)\,dx = 1 \Rightarrow \begin{cases} \text{box} \quad A^2\int_0^a \sin^2 k_n x\,dx = A^2\dfrac{a}{2} \Rightarrow A = \sqrt{\dfrac{2}{a}} \\ \text{circle} \quad A = 1 \end{cases}$$

Integrated over all possible position the particle can take.

The wave function contains the information about the location of the particle ($P(x) = |\psi(x)|^2$) ... but this is not all! Actually $\psi(x)$ contains information about <u>all</u> the properties the particle has.

This is the way to extract this properties, as suggested by de-Broglie intuition:

de Broglie

$\psi_{dB} = \sin(kx - \omega t)$

momentum $p = \frac{h}{\lambda} = \hbar k$

energy $E = h\nu = \hbar\omega$

Wave-function

$\psi = e^{i(kx - \omega t)}$

$p \to -i\hbar\frac{\partial}{\partial x}$

[in fact $-i\hbar\frac{\partial}{\partial x}\psi = (\hbar k)\psi$]

$E \to +i\hbar\frac{\partial}{\partial t}$ [in fact

$i\hbar\frac{\partial}{\partial t}\psi = (\hbar\omega)\psi$]

Thus the wave-function ψ describes a state of de<u>finite</u> energy or momentum if the action of $i\hbar\frac{\partial}{\partial t}$ or $-i\hbar\frac{\partial}{\partial x}$ leaves the form of ψ invariant and multiplies it by a constant (which <u>is</u> the value of the momentum or energy).

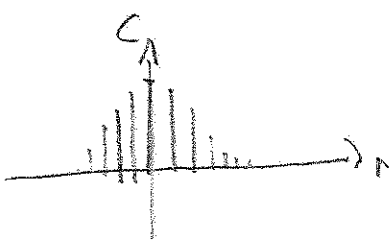
This is called eigenvalue equation:

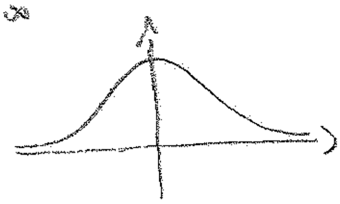
$$\underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)}_{\text{Operator}}\underbrace{\left[e^{i(kx-\omega t)}\right]}_{\psi(x,t)} = \underbrace{(\hbar k)}_{\text{Eigenvector}}\underbrace{\left[e^{i(kx-\omega t)}\right]}_{\substack{\psi(x,t)\\ \text{Eigenstate}}}$$

Of course a wave-function describing a state of definite momentum corresponds to a particle completely delocalized [$P(x)$ is constant !] as consequence of the uncertainty principle.

So let us build a wave packet (i.e. a superposition of many plane waves (let us focus our attention on the time $t=0$)

$$\sum c(k)\, e^{ikx} \rightarrow \int_{-\infty}^{\infty} c(k)\, e^{ikx} = \psi(x)$$





This $\psi(x)$ does not describe a particle with definite momentum